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FLOER HOMOLOGY FOR CONNECTED SUMS OF HOMOLOGY 3-SPHERES

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1. Introduction

Floer homology is a mod 8-graded homology theory for integral homology 3-spheres Y [8]. "The Floer" chain groups $C_*(Y)$ are generated by irreducible SU(2)-representations of the fundamental group $\pi_1(Y)$. The boundary operator ∂, however, depends on the nonlinear analysis; namely, the 1-dimensional moduli spaces on $Y \times \mathbf{R}$. Floer proved that $\partial^2 = 0$, the homology of the chain complex $HF_*(Y) = H_*(C_*(Y), \partial)$ is independent of metrics and perturbations, and the Euler characteristic of Floer homology is twice Casson's invariant [8], [14]. Floer homology also provides a natural setting for relative Donaldson invariants of smooth 4-manifolds with boundary (see [1]). For a number of related recent developments in geometry and topology, see, for example, [7].

The first calculations of Floer homology were carried out by Fintushel and Stern, who computed the Floer homology for Brieskorn spheres. Their calculations are based on the evenness of the spectral flow for the Brieskorn spheres, where the trivial boundary map states that the Floer homology is the same as the Floer chain complex. Our goal is to understand how to calculate the Floer homology for the connected sum of two homology 3-spheres.

This paper is based on understanding the boundary map for the Floer homology of $Y_0\#Y_1$, i.e., the 1-dimensional moduli spaces on the connected sum $(Y_0\#Y_1) \times \mathbf{R}$ where Y_i is a homology 3-sphere for $i = 0, 1$. This relies on the (Taubes) glueing procedure on a noncompact 4-manifold with almost harmonic 2-forms in the glueing region. Given anti-self-dual connections A_i on $Y_i \times \mathbf{R}$, we deform them slightly and then glue them together to form an "almost anti-self-dual" connection $A_0\#A_1$ on $(Y_0 \times Y_1) \times \mathbf{R}$. The problem is then to deform $A_0\#A_1$ into an anti-self-dual connection. The "almost harmonic" 2-forms arise from the pullback of the volume form on the central S^2 multiplied by cutoff functions. The

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difficulty in this type of glueing procedure is that the grafting techniques on compact 4-manifolds cannot be applied directly, even for a careful choice of a weighted Sobolev space.

Our approach to overcome this difficulty is similar to that of Donaldson and Sullivan in [5]. It is based on a construction of the bounded right inverse of $d_A^+ = \frac{1+*}{2} d_A$ on the product of a tiny annulus with $\mathbf{R}$ in a weighted Sobolev space with respect to the "merged" metric on $(Y_0 \# Y_1) \times \mathbf{R}$ (see §3.3 for details). A careful choice of this annulus is necessary in order to compare the anti-self-duality operators with respect to different metrics in the overlap region. We have a uniformly bounded right inverse of $d_{A_i}^+$ on $Y_i \times \mathbf{R}$. In order to get a uniformly bounded right inverse for the anti-self-duality operator with the twist $A_0 \# A_1$, we first obtain the uniformly bounded right inverse for the almost anti-self-dual connection on $Y_i \times \mathbf{R}$, and then patch them together by using the fact that the original metric and the merged metric on the annulus region are very close. In this way we can estimate the difference of the two operators with different metrics and get the uniformly bounded inverse for $d_{A_0 \# A_1}^+$. Because of the operator d_A^+, unlike d_A^*, involves no derivative of the metric, C^0-close metrics are sufficient for the estimates. Finally we prove the glueing theorem and the splitting theorem for anti-self-dual connections on $(Y_0 \# Y_1) \times \mathbf{R}$ by constructing a parametrized bounded right inverse on $(Y_0 \# Y_1) \times \mathbf{R}$, and the inverse function theorem.

Using the grafing theorem and the spectral flow calculations, we will show that the glueing parameter spaces form a filtration for the Floer chain groups of $Y_0 \# Y_1$. Hence we have a spectral sequence for the Floer homology of $Y_0 \# Y_1$, and the spectral sequence and its differentials are given in [10].

This paper is organized as follows. In §2 we give a review of Floer homology and study the spectral flow. We present the main analytic part for the boundary map of Floer homology of $Y_0 \# Y_1$ in §3.

2. Floer homology of homology 3-spheres

2.1. Floer homology.

In this subsection, we will give a brief description of gauge theory on 3-manifolds and review the definition of Floer homology. For details see [3], [6], and [8].

Let Y be a homology 3-sphere, i.e., an oriented closed 3-dimensional smooth manifold with $H_1(Y, Z) = 0$, and let $P \to Y$ be a smooth principal SU(2)-bundle. (Since $c_2(P) = 0$, this bundle is trivial.) Fix a trivialization $Y \times SU(2)$ of P and let θ be the associated trivial connection. Denote the Sobolev L_k^p-space of connections on P by $\mathscr{A}(P)$. This space has a natural affine structure with underlying vector space $\Omega^1(Y, \operatorname{ad} P)$, where $\operatorname{ad} P$ is the adjoint bundle. $\mathscr{A}(P)$ is acted upon by the gauge group $\mathscr{G}$ of L_{k+1}^p-automorphisms of P, and the orbit space $\mathscr{B}(P) = \mathscr{A}(P)/\mathscr{G}$ is well-defined when $k+1 > 3/p$. The irreducible connections form an open dense subspace $\mathscr{B}^*(P)$ of $\mathscr{B}(P)$ which is a Banach manifold with

$$T_a\mathscr{B}^*(P) \equiv \{\alpha \in L_k^p(\Omega^1(Y, \operatorname{ad} P)) | d_a^*\alpha = 0\},$$

where d_a^* is the L^2-adjoint of d_a (covariant derivative on sections of $\operatorname{ad} P$) with respect to some metric on Y.

The Chern-Simons functional $\operatorname{cs}: \mathscr{A}(P) \to \mathbf{R}$ is defined as

$$\operatorname{cs}(a) = \frac{1}{2}\int_Y \operatorname{tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a),$$

and satisfies $\operatorname{cs}(g\cdot a) = \operatorname{cs}(a) + 2\pi \deg(g)$ for gauge transformations $g: Y \to SU(2)$. Thus cs is well-defined on $\widetilde{\mathscr{B}}(P) = \mathscr{A}(P)/\{g \in \mathscr{G}: \deg(g) = 0\}$, and it descends to a function $\operatorname{cs}: \mathscr{B}(P) \to R/2\pi Z$ which plays the role of a Morse function in defining Floer homology. Furthermore the differential of cs is

$$d\operatorname{cs}(a)(\alpha) = \int_Y \operatorname{tr}(F_a \wedge \alpha),$$

hence its critical set consists of the flat connections $\mathscr{R}(\mathscr{B}(P)) = \{a \in \mathscr{B}(P) | F_a = 0\}$. (Here F_a is the curvature 2-form on Y.) It is well-known that elements of $\mathscr{R}(\mathscr{B}(P))$ are in 1-1 correspondence with those of

$$\mathscr{R}(Y) = \operatorname{Hom}(\pi_1(Y), SU(2))/\operatorname{ad} SU(2),$$

the SU(2)-representations of $\pi_1(Y)$ modulo conjugacy. Given any metric on Y, the Hodge star operator applied to the curvature F_a gives a vector field

$$f(a) = *F_a \in L_{k-1}^p(\Omega^1(Y, \operatorname{ad} P)).$$

Comparing with $T_a\mathscr{B}^*(P)$, we note the different Sobolev norm and denote the latter by $\mathscr{L}_a$. Hence f is a section of the bundle with fiber $\mathscr{L}_a$. A representation $a \in \mathscr{R}(Y)$ is said to be *nondegenerate* if the twisted cohomology $H^1(Y; \operatorname{ad} \alpha)$ vanishes. Note that this is the same as requiring that $\ker df(a) = \ker *d_a = 0$, where $*d_a$ is the Hessian of the Chern-Simons functional.

A 1-parameter family $\{a(t) | t \in \mathbf{R}\}$ of connections on P gives rise to a connection A with vanishing t-component on the trivial SU(2) bundle over $Y \times \mathbf{R}$. Floer's crucial observation is that trajectories of the vector field f, i.e., the flow lines of

$$\frac{\partial a}{\partial t} + f(a(t)) = 0 \quad \text{or} \quad \frac{\partial a}{\partial t} = *F(a(t)),$$

can be identified with instantons A on $Y \times \mathbf{R}$, and $A|_{Y\times\{t\}} = a(t)$. A trajectory flow "connects" two flat connections on Y if and only if the Yang-Mills energy of the trajectory (as a connection on $Y \times \mathbf{R}$ with trivial component in the $\mathbf{R}$ direction) is finite. One needs to show that all zeros of f are nondegenerate and that their stable and unstable manifolds intersect transversally in smooth finite dimensional manifolds. Floer has shown that one can perturb the Chern-Simons functional to achieve this (see [8]). For the rest of this paper, we assume that the Chern-Simons functional has been so perturbed, so that all irreducible representations are isolated and nondegenerate. Since $\mathscr{R}(Y)$ is compact, they are also finite.

For the analysis of ASD connections, it is convenient to work with the weighted Sobolev space $L^p_{k,\delta}$ that we will introduce in §3. For each connection A the anti-self-duality operator induces a Fredholm operator

$$d_A^* \oplus d_A^+ : L^p_{k+1,\delta}(\Omega^1(Y \times \mathbf{R}, \operatorname{ad} P)) \to L^p_{k,\delta}((\Omega^0 \oplus \Omega^2_+)(Y \times \mathbf{R}, \operatorname{ad} P)).$$

We say that A is *regular* if $d_A^* \oplus d_A^+$ is surjective. In terms of the complex we have

$$L^p_{k+1,\delta}(\Omega^0(Y \times \mathbf{R}, \operatorname{ad} P)) \xrightarrow{d_A} L^p_{k,\delta}(\Omega^1(Y \times \mathbf{R}, \operatorname{ad} P))$$
$$\xrightarrow{d_A^+} L^p_{k-1,\delta}(\Omega^2_+(Y \times \mathbf{R}, \operatorname{ad} P)).$$

The regularity of A means that $H^0_A = 0$ (irreducible) and $H^2_A = 0$ (generic). For a nondegenerate critical point α of cs, the spectral flow is $SF(\alpha, \theta) = \operatorname{Index}(d_A^* \oplus d_A^+)(\alpha, \theta)$, the Atiyah-Patodi-Singer index of the anti-self-duality operator over $Y \times \mathbf{R}$. So

$$\mu(\alpha) \equiv \operatorname{Index}(d_A^* \oplus d_A^+)(\alpha, \theta) \pmod 8,$$

where A is any family of connections $\{a(t)\} \in \mathscr{B}(P)$ over Y with $a(+\infty) = \theta$, $a(-\infty) = a_\alpha$ (see [8]). Floer's chain group $C_j(Y)$ is defined to be the free module generated by the irreducible flat connections α with $\mu(\alpha) = j \pmod 8$.

Note. Changing the orientation of Y switches the sign of cs and hence the spectrum of the Hessian reverses, so

$$-\mu_{-Y}(\alpha) = 3 - (-\mu_Y(\alpha)) \pmod 8,$$

i.e., $\mu_{-Y}(\alpha) = 5 - \mu_Y(\alpha) \pmod 8$.

Define $\mathscr{M}_{Y\times\mathbf{R}}$ to be the moduli space of finite-energy ASD connections on $Y\times\mathbf{R}$, and let $\mathscr{M}(\alpha, \beta)$ be the subspace of those A such that $\lim_{t\to-\infty} A = \alpha$, $\lim_{t\to+\infty} A = \beta$ for fixed flat connections α and β. $\mathscr{M}(\alpha, \beta)$ is a smooth, canonically oriented manifold which has dimension congruent to $\mu(\alpha) - \mu(\beta)$ (mod 8). The moduli space $\mathscr{M}(\alpha, \beta)$ has finitely many connected components each of which admits a proper, free $\mathbf{R}$-action arising from the translations in $Y\times\mathbf{R}$. If $\mu(\alpha)-\mu(\beta) = 1$ (mod 8), let $\mathscr{M}^1(\alpha, \beta)$ be the union of 1-dimensional components of $\mathscr{M}(\alpha, \beta)$. Further perturbations make all the $\mathscr{M}^1(\alpha, \beta)$ regular. Then $\mathscr{M}^1(\alpha, \beta)/\mathbf{R}$ will be a compact oriented 0-manifold, i.e., it is a finite set of signed points. The differential $\partial\colon C_j \to C_{j-1}$ of Floer's chain complex is defined by

$$\partial\alpha = \sum_{\beta\in C_{j-1}} \#\widehat{\mathscr{M}}(\alpha, \beta)\beta, \tag{1}$$

where $\widehat{\mathscr{M}}(\alpha, \beta) = \mathscr{M}^1(\alpha, \beta)/\mathbf{R}$, $\#\widehat{M}(\alpha, \beta)$ is the algebraic number of points, and the sign is given by the spectral flow. Floer has shown that $\partial^2 = 0$, hence $\{C_j, \partial)_{j\in Z_8}$ is a chain complex graded by $\mathbf{Z}_8$. The homology of this complex is called the Floer homology, denoted by HF_j. Floer has shown that it is independent of the choice of metric on Y and of perturbations (see [3], [8]).

2.2. The spectral flow on connected sums.

The connected sum $Y = Y_0\#Y_1$ of two homology 3-spheres is again a homology 3-sphere, whose fundamental group $\pi_1(Y_0\#Y_1)$ is the free product of $\pi_1(Y_0)$ and $\pi_1(Y_1)$. There are four types of SU(2) representations of $\pi_1(Y_0\#Y_1)$:

(1) $\theta = \theta_0\#\theta_1$,
(2) $\theta_0\#\alpha_1$,
(3) $\alpha_0\#\theta_1$,
(4) $\alpha_0\#\alpha_1$,

where the α_i are irreducible representations of $\pi_1(Y_i)$, and θ_i is the trivial representation of $\pi_1(Y_i)$, $i = 0, 1$. These four types of representations correspond to equivalence classes of flat connections glued together by the standard glueing construction [4]. In each case we have a family $a_0\#a_1$ of flat connections parametrized by a copy of SU(2), which can be identified with the automorphisms of a fiber over a point in the gluing region. Two elements of this family corresponding to automorphisms ρ_0, ρ_1 are gauge equivalent if and only if $\rho_0\rho_1^{-1}$ extends to an element of the isotropy group Γ_{a_0} or Γ_{a_1}. Thus the corresponding family of gauge equivalence classes is

$SU(2)/\Gamma_{a_0} \times \Gamma_{a_1}$. Since $\Gamma_\theta \equiv SU(2)$ and $\Gamma_a = Z_2$ for a irreducible, the first three types of representations give rise to a unique gauge equivalence class, whereas the last type of representation gives a copy of SO(3) for each pair of irreducible representations. In §3 we show that all trajectories between these representations are obtained by grafting together existing trajectories from both sides. Thus one needs to compute the spectral flow along such trajectories.

The spectral flow $SF(\alpha, \theta)$ is (modulo 8) the index of the Fredholm operator $D_A = d_A^{*\delta} \oplus d_A^+$ on the weighted Sobolev space with sufficiently small weight δ. Then the Floer grading is given by

$$\mu(\alpha) \equiv \operatorname{index} D_A(\alpha, \theta) \pmod 8. \tag{2}$$

One can consider the calculation of the index of the anti-self-duality operator as a boundary value problem with Atiyah-Patodi-Singer global boundary conditions [2]. We have

$$\operatorname{Index}(d_A^{*\delta} \oplus d_A^+)(\alpha, \beta) = -2\int_{Y\times\mathbf{R}} p_1(A) - \frac{h_\beta + \rho_\beta(0)}{2} + \frac{-h_\alpha + \rho_\alpha(0)}{2} \tag{3}$$

where $p_1(A)$ is the Pontryagin form, the term h_β is the sum of the dimensions of $H^i(Y, \operatorname{ad}\beta)$, $i = 0, 1$, and ρ_β is the ρ-invariant of the signature operator $*d_{\alpha_\beta} - d_{\alpha_\beta}*$ over Y restricted to even forms (cf. [6]). An application of the signature formula to $Y \times I$ shows that $\rho_\alpha = \rho_\alpha(0)$ is independent of the Riemannian metric on Y and is an orientation-preserving diffeomorphism invariant of Y and α.

Lemma 2.2.1. *For $\alpha_i \in \mathbf{R}(Y_i)$ irreducible, we have*

(1) $\rho_{\alpha_0\#\alpha_1}(0) = \rho_{\alpha_0}(0) + \rho_{\alpha_1}(0)$,

(2) $h_{\alpha_0\#\alpha_1} = h_{\alpha_0} + h_{\alpha_1} + 3$, $h_{\alpha_0\#\theta_1} = h_{\alpha_0}$, $h_{\theta_0\#\alpha_1} = h_{\alpha_1}$, $h_{\theta_0\#\theta_1} = 3$.

Proof. (1) Consider the cobordism X built by attaching a 1-handle to $(Y_0 \coprod Y_1) \times \{1\}$ in $(Y_0 \coprod Y_1) \times I$. The boundary of X is $Y_0\#Y_1 \coprod -Y_0 \coprod -Y_1$ (note that $\pi_1(X) = \pi_1(Y_0\#Y_1)$). Then there are natural inclusions $\mathscr{R}(Y_i) \to \mathscr{R}(Y_0\#Y_1)$ such that the pair (α_0, α_1) can be extended to a unitary representation of $\pi_1(Y_0\#Y_1)$. (In fact, if the α_i are both irreducible, then there is an SO(3)-family of such extensions.) By Theorem 2.4 in [2], we have

$$\rho_{\alpha_0\#\alpha_1}(Y_0\#Y_1) - \rho_{\alpha_0\#\alpha_1}(Y_0 \coprod Y_1) = 3\operatorname{sign}(X) - \operatorname{sign}_{\alpha_0\#\alpha_1}(X),$$

where $H^2(X) = 0$ and $H^2(X; \operatorname{ad}\alpha) = 0$. So we get the signatures satisfying $\operatorname{sign}(X) = 0$ and $\operatorname{sign}_{\alpha_0\#\alpha_1}(X) = 0$. Thus $\rho_{\alpha_0\#\alpha_1}(Y_0\#Y_1) = \rho_{\alpha_0}(Y_0) + \rho_{\alpha_1}(Y_1)$.

(2) Since α_0, α_1 are irreducible, the Betti numbers $h^0_{\alpha_0}$, $h^0_{\alpha_1}$, and $h^0_{\alpha_0\#\alpha_1}$ vanish. The Mayer-Vietoris sequence

$$0 \to H^0(S^2, \operatorname{ad} SU(2)) \to H^1_{\alpha_0\#\alpha_1}(Y_0\#Y_1, \operatorname{ad} SU(2))$$
$$\to H^1_{\alpha_0}(Y_0, \operatorname{ad} SU(2)) \oplus H^1_{\alpha_1}(Y_1, \operatorname{ad} SU(2)) \to 0$$

then shows that $h_{\alpha_0\#\alpha_1} = h_{\alpha_0} + h_{\alpha_1} + 3$.

Clearly $h_{\theta_0\#\theta_1} = 3$, so we consider the case of θ_0 and α_1, where α_1 is irreducible. We have $h^0_{\theta_0\#\alpha_1} = 0$ and $h^1_{\theta_0} = 0$. Again applying the Mayer-Vietoris sequence

$$0 \to H^0_{\theta_0}(Y_0, \operatorname{ad} SU(2)) \oplus H^0_{\alpha_1}(Y_1, \operatorname{ad} SU(2)) \to H^0(S^2, \operatorname{ad} SU(2))$$
$$\to H^1_{\theta_0\#\alpha_1}(Y_0\#Y_1, \operatorname{ad} SU(2)) \to H^1_{\theta_0}(Y_0, \operatorname{ad} SU(2)) \oplus H^1_{\alpha_1}(Y_1, \operatorname{ad} SU(2))$$
$$\to 0$$

and using $h^1_{\theta_0} = 0$, we have

$$(3+0) - 3 + h^1_{\theta_0\#\alpha_1} - (0 + h^1_{\alpha_1}) = 0,$$

i.e., $h_{\theta_0\#\alpha_1} = h_{\alpha_1}$.

Lemma 2.2.2. *For irreducible representations $\alpha_i \in \mathscr{R}(Y_i)$, we have the following addition properties for the Floer grading μ:*

$$\mu(\alpha_0\#\alpha_1) = \mu(\alpha_0) + \mu(\alpha_1),$$
$$\mu(\theta_0\#\alpha_1) = \mu(\alpha_1);\ \mu(\alpha_0\#\theta_1) = \mu(\alpha_0).$$

Proof. To compute $\mu(\alpha_i)$ we can use any connections A_i over $Y_i \times \mathbf{R}$ which interpolate between θ_i and α_i. We choose A_i to be flat in the regions $B^3 \times \mathbf{R}$ used to make the connected sum $(Y_0\#Y_1) \times \mathbf{R}$. So the A_i's match to give a connection $A_1\#A_2$ over $(Y_0\#Y_1) \times \mathbf{R}$ which interpolates from $\theta_0\#\theta_1$ to $\alpha_0\#\alpha_1$. By (2) and (3) we have

$$\mu(\alpha_0\#\alpha_1) = -2\int_{Y\times R} p_1(A_1\#A_2) - \frac{h_{\alpha_0\#\alpha_1} - \rho_{\alpha_0\#\alpha_1}}{2} - \frac{h_{\theta_0\#\theta_1} + \rho_{\theta_0\#\theta_1}}{2} \pmod 8,$$

where $Y = Y_0\#Y_1$. From our choice of A_i,

$$p_1(A_1\#A_2) = p_1(A_1) + p_1(A_2).$$

Since $\rho_{\theta_0\#\theta_1} = 0$ and $\rho_{\theta_i} = 0$, our result follows from Lemma 2.2.1. Similarly one checks that $\mu(\theta_0\#\alpha_1) = \mu(\alpha_1)$ and $\mu(\alpha_0\#\theta_1) = \mu(\alpha_0)$. q.e.d.

In a similar manner one shows

Proposition 2.2.3. *If* $\alpha_i \in \mathscr{R}^*(Y_i)$ *and at least one* $\beta_i \in \mathscr{R}^*(Y_i)$, *then*

(4) $\operatorname{Index} D_A(\alpha_0\#\alpha_1, \beta_0\#\beta_1) = \operatorname{Index} D_A(\alpha_0, \beta_0) + \operatorname{Index} D_A(\alpha_1, \beta_1) + 3,$

and therefore

(5) $\dim \mathscr{M}(\alpha_0\#\alpha_1, \beta_0\#\beta_1) = \dim \mathscr{M}(\alpha_0, \beta_0) + \dim \mathscr{M}(\alpha_1, \beta_1) + 3.$

The '3' in this formula arises from the glueing parameters: $\alpha_0\#\alpha_1$ is a family of connections parametrized by $\rho_0 \in SO(3)$, $\beta_0\#\beta_1$ is in turn parametrized by $\rho_1 \in SO(3)$, and the pairs $(\alpha_0\#\alpha_1, \beta_0\#\beta_1)$ are gauge equivalent when $\rho_0 = \rho_1$. Thus the glueing parameters give three extra dimensions on the left-hand side of equation (5). In [10] we will show that perturbing the Chern-Simons functional breaks the SO(3) symmetry, splitting $\mathscr{M}(\alpha_0\#\alpha_1, \beta_0\#\beta_1)$ into a finite union of moduli spaces with dimension three less than that of $\mathscr{M}(\alpha_0\#\alpha_1, \beta_0\#\beta_1)$. Once this perturbation is done, we can apply Floer theory as described at the beginning of this section.

We can now see which trajectory flows contribute to the Floer boundary (1) on the connected sum $Y_0\#Y_1$ of homology spheres. By definition, the only relevant trajectories are those that lie in a 1-dimensional component $\mathscr{M}_i$ of $\mathscr{M}(\alpha_0\#\alpha_1, \beta_0\#\beta_1)$ (after perturbation). By (5) and the above remarks there is a possibility of having

$$(6) \qquad \begin{aligned} 1 = \dim \mathscr{M}_i &= \dim \mathscr{M}(\alpha_0\#\alpha_1, \beta_0\#\beta_1) - 3 \\ &= \dim \mathscr{M}(\alpha_0, \beta_0) + \dim \mathscr{M}(\alpha_1, \beta_1), \end{aligned}$$

so the moduli spaces $\mathscr{M}(\alpha_i, \beta_i)$ have dimension 0 or 1. Thus the boundary operator can be computed once we understand the moduli spaces obtained by glueing a 1-dimensional moduli space on one side of $Y_0\#Y_1$ to a 0-dimensional moduli space (i.e., a constant flat connection) on the other.

3. Grafting

The essential step in the calculation of the Floer homology of a connected sum of homology 3-spheres Y_0, Y_1 is to understand the structure of the 1-dimensional moduli space of ASD connections on $(Y_0\#Y_1) \times \mathbf{R}$. The structure relies on grafting together ASD connections on noncompact 4-manifolds. The major problem is the existence of harmonic 2-forms in the glueing region. It is difficult to obtain estimates in the overlap region relating the "merged" metric with the original metrics g_i on Y_i. For the merged metric g we will take a weighted average. The usual Rayleigh quotient for the first eigenvalue involves the operator d^{*g}, and in order to get a uniform bound on the first eigenvalue on the connected sum from the first eigenvalues on both sides, we have to compare d^{*g} and d^{*g_i}. These operators involve the derivative term of the weighted average metric with no control of the glueing parameter ε (the "neck-length"). Thus we adopt Donaldson and Sullivan's technique for building a right inverse directly (cf. [5]).

We begin by looking at a special feature of the $\mathbf{R}$-action on the equivalence classes of connections which will help us solve the anti-self-duality equation

$$F_A^+ + (d_A^+ + d_A^{*\delta})a + a \wedge a = 0$$

uniquely on the subspace of $\Omega^1_{\text{ad}}(Y \times \mathbf{R})$, which is perpendicular to H^1_A. Then we show that for all balanced 1-dimensional ASD connections on $Y_i \times \mathbf{R}$, there is a uniform lower eigenvalue. Using the parametric method to construct the right inverse on the connected sum and applying the inverse function theorem, we are able to prove a glueing and splitting theorem for 1-dimensional ASD connections over $(Y_0 \# Y_1) \times \mathbf{R}$ (Theorem 3.3.10).

Throughout this section we assume that the anti-self-duality operator is regular. (As we have mentioned above, this can always be achieved by a compact perturbation of the anti-self-duality operator. For the sake of simplicity we shall ignore the perturbation.)

3.1. Properties of balanced connections.

Let Y be a closed, connected, oriented, smooth homology 3-sphere. For $\delta \geq 0$ (to be determined), let $e_\delta \colon Y \times \mathbf{R} \to \mathbf{R}$ be a smooth positive function with $e_\delta(y, t) = e^{\delta|t|}$ for $|t| \geq 1$. Let E be an SU(2)-vector bundle over $Y \times \mathbf{R}$ with a translationally invariant metric and a metric-preserving connection. To define Banach manifolds $\mathscr{B}(a, b)$ of paths connecting two flat connections a and b in $\mathscr{B}_Y$, choose any smooth representatives $a, b \in \mathscr{A}_Y$ and a smooth connection C on $Y \times \mathbf{R}$ which coincides with a for $t \leq -1$ and with b for $t \geq 1$. Define the $L^p_{k,\delta}$-norm on sections u of E by

$$\|u\|_{L^p_{k,\delta}} = \|e_\delta \cdot u\|_{L^p_k}. \tag{7}$$

Then

$$\mathscr{A}_\delta(a, b) = C + L^p_{1,\delta}(\Omega^1_{\text{ad}}(Y \times \mathbf{R}))$$

is an affine space and is independent of the choice of C. The corresponding gauge group is the group $\mathscr{G}^p_{2,\delta}$ obtained by completing the compactly supported gauge transformations in the $L^p_{2,\delta}(\Omega^0_{\text{ad}}(Y \times \mathbf{R}))$. We need $p > 2$ to construct the orbit space $\mathscr{B}_{Y \times \mathbf{R}} = \mathscr{A}_\delta^{1,p}/\mathscr{G}_\delta^{2,p}$.

Proposition 3.1.1. (1) *Let*

$$D_a\colon L^p_{1,\delta}(\Omega^1\oplus\Omega^0)(Y, \operatorname{ad}\mathrm{SU}(2)) \to L^p_{0,\delta}(\Omega^1\oplus\Omega^0)(Y, \operatorname{ad}\mathrm{SU}(2))$$

be the operator $D_a(\alpha, \beta) = (*d_a\alpha - d_a\beta, -d_a^*\alpha)$. *Then there exists a positive* λ_0 *such that for all* $a \in \mathscr{R}^*(Y)$ *the eigenvalues of* D_a *satisfy* $|\lambda(D_a)| \ge \lambda_0$.

(2) *If* $F(A)$ *is in* L^p *for* $p \ge 2$, *then there is a constant* C_A *such that*

$$\sup|F_A|_{y,t} \le C_A e^{-\gamma|t|}, \tag{8}$$

where $\gamma = \gamma(\lambda_0) > 0$, *and* C_A *is continuous in* A.

Proof. The first statement is from [8], and the second is in [3] (see 4.1). q.e.d.

Fix a positive $\delta < \min\{\lambda_0, \gamma/2\}$. We will henceforth use the norm (7) with this δ. Let us denote $\|u\|_{L^p_{1,\delta}(A)} = \|\nabla_A u\|_{L^p_{0,\delta}} + \|u\|_{L^p_{0,\delta}}$ (and $\|u\|_{L^p_{1,\delta}} = \|u\|_{L^p_{1,\delta}(C)}$).

Definition 3.1.2. The *balancing function* $b\colon \mathscr{B}_{Y\times\mathbf{R}} \to \mathbf{R}$ is given by the equation

$$\int_{-\infty}^{b(A)} \|F(A)\|^2_{L^2(Y)} = \int_{b(A)}^{\infty} \|F(A)\|^2_{L^2(Y)}.$$

(So the value $b(A)$ is the time which splits the action of A in half.)

Lemma 3.1.3. (1) *Shifting the connection* A *in the t-direction,* $A(t) \to A(t \pm s)$, *one has*

$$b(A(t+s)) = b(A(t)) - s, \qquad b(A(t-s)) = b(A(t)) + s.$$

(2) *Let* $\mathscr{B}_0 = b^{-1}(0)$ *be the space of equivalence classes of connections whose actions are balanced at* 0. *Then there is a one-to-one map from* $\mathscr{B}_0$ *to* $\mathscr{B}_s = b^{-1}(s)$ *for any* $s \in \mathbf{R}$.

(3) *If* A *is not a constant flat connection, the derivative of* b *is*

$$D_A b(a) = \int_{-\infty}^{+\infty}\left\langle \frac{\operatorname{sing}(t - b(a))}{\|F(A)\|^2_{L^2(Y,\mathbf{R})}} d_A^* F_A, a\right\rangle.$$

Proof. (1) is proved by a change of variable. (2) follows from (1). For (3) we note that

$$\int_{-\infty}^{b(A+sa)} \|F(A+sa)\|^2_{L^2(Y)} = \int_{b(A+sa)}^{+\infty} \|F(A+sa)\|^2_{L^2(Y)}.$$

Taking the derivative with respect to s at $s = 0$ and combining the terms, one has

$$\|F(A)\|^2_{L^2(Y\times\mathbf{R})} D_A b(a) = \int_{b(A)}^{+\infty} \langle d_A^* F_A, a\rangle - \int_{-\infty}^{b(A)} \langle d_A^* F_A, a\rangle.$$

Now $\|F(A)\|^2_{L^2(Y\times\mathbf{R})} = 0$ if and only if $-(\partial a/\partial t)dt + F_a = 0$, i.e., if and only if A is a constant flat connection, contrary to our hypothesis. Thus (3) follows.

Definition 3.1.4. Set the *balanced moduli space* $\mathscr{M}^{\text{bal}}_{Y\times\mathbf{R}} = \{A \in \mathscr{M}_{Y\times\mathbf{R}} \subset \mathscr{B}_{Y\times\mathbf{R}} | b(A) = 0\}$.

Lemma 3.1.5. *For* $A \in \mathscr{M}^{\text{bal}}_{Y\times\mathbf{R}}$, $y \in Y$, *and each* $p \geq 2$, *there exist constants* M_0, C_1, C_2 *independent of* A *such that the following hold:*

(i) *If* $\dim \mathscr{M}_{Y\times\mathbf{R}} \leq 1$, *then* $\mathscr{M}^{\text{bal}}_{Y\times\mathbf{R}}$ *is compact, and*

$$\int_{Y\times\mathbf{R}} e^{p\delta|t|}|F_A|^p < C_1, \quad \int_{B^3_y(\varepsilon)\times\mathbf{R}} e^{p\delta|t|}|F_A|^p < C_2\varepsilon^3. \tag{9}$$

(ii) *If* $\dim \mathscr{M}_{Y\times\mathbf{R}} < 8$, *then* $\|F_A\|_{L^\infty(Y\times\mathbf{R})} \leq M_0$.

Proof. (i) No sequence of connections in $\mathscr{M}^{\text{bal}}_{Y\times\mathbf{R}}$ can converge weakly to a limit plus an instanton bubble, since bubbling needs $\dim \mathscr{M}_{Y\times\mathbf{R}} \geq 8$. There is only one other way for a sequence in $\mathscr{M}^{\text{bal}}_{Y\times\mathbf{R}}$ to fail to have a convergent subsequence; it may have a subsequence $\{A_n\}$ converging weakly to a disjoint union of connections $A_{-\infty} \in \mathscr{M}_{Y\times\mathbf{R}}(a, b)$, $A_0 \in \mathscr{M}_{Y\times\mathbf{R}}(b, c)$, $A_{+\infty} \in \mathscr{M}_{Y\times\mathbf{R}}(c, d)$ where a, b, c, d denote the limits and at least one of $A_{-\infty}$, $A_{+\infty}$ is not constant flat (otherwise $\{A_n\}$ actually converges to A_0). If, say, $A_{+\infty}$ is not constant flat, then $\dim \mathscr{M}_{Y\times\mathbf{R}}(c, d) \geq 1$. Since each A_n is balanced, the limit $A_{-\infty} \coprod A_0 \coprod A_{+\infty}$ is also balanced, and it follows that $\dim \mathscr{M}_{Y\times\mathbf{R}}(a, b) + \dim \mathscr{M}_{Y\times\mathbf{R}}(b, c) \geq 1$. This is impossible since the dimension of the moduli space $\mathscr{M}_{Y\times\mathbf{R}}(a, d)$ which contains the A_n is equal to 1. Thus $\mathscr{M}^{\text{bal}}_{Y\times\mathbf{R}}$ is compact.

The constant C_A in (8) is continuous in A, so by compactness it can be chosen uniform for A in $\mathscr{M}^{\text{bal}}$, and the bounds (9) follow.

(ii) Suppose not. Then there exists a sequence $\{A_n\} \in \mathscr{M}_{Y\in\mathbf{R}}$ with $\|F_{A_n}\|_{L^\infty(Y\times\mathbf{R})} > n$. Thus we have (y_n, t_n) such that $|F_{A_n}|_{(y_n, t_n)} = n$. Let $A'_n = A_n(t - t_n)$ (rescaling). Then $|F_{A'_n}|_{(y_n, 0)} = n$. The application of Uhlenbeck's compactness theorem to the compact space $Y \times [-1, 1]$ shows that there exists a subsequence $\{A_i\}$ with a bubble point, and this requires $\dim \mathscr{M}_{Y\times\mathbf{R}} \geq 8$, contradicting our assumption.

Proposition 3.1.6. *The space* $\mathscr{B}^{\text{bal}}_{Y\times\mathbf{R}} = \{A \in \mathscr{B}_{Y\times\mathbf{R}} | b(A) = 0\}$ *of balanced connections is a smooth manifold with codimension* 1, *and the moduli space* $\mathscr{M}^1_{Y\times\mathbf{R}}$ *is transversal to* $\mathscr{B}^{\text{bal}}_{Y\times\mathbf{R}}$.

Proof. Since an arbitrary $A' \in \mathscr{M}^1_{Y\times\mathbf{R}}$ is not a constant flat connection, it has a translate A under the $\mathbf{R}$-action which lies in $\mathscr{B}^{\text{bal}}_{Y\times\mathbf{R}}$. Note that

$\|F_A\|^2 \neq 0$. Let $A = a(t)$. If $0 = d_A^* F_A = -(d_a *(\partial a/\partial t) + \partial * F_a/\partial t) \wedge dt + d_a * F_a$, then we get $*F_a = 0$. Since A is anti-self-dual, $\partial a/\partial t = *F_a = 0$ and A is constant flat connection. But this is not true, so the L^2-normal vector

$$v = \frac{\mathrm{sign}(t) \cdot d_A^* F_A}{\|F_A\|^2}$$

to $T\mathscr{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ at A is nontrivial. By the implicit function theorem, $\mathscr{B}^{\mathrm{bal}}_{Y\times\mathbf{R}} = b^{-1}(0)$ is a smooth codimension 1 Banach submanifold, and $D_A b\colon T_A N \to T_0\mathbf{R}$ is an isomorphism where $T_A N$ is the subspace of $T_A\mathscr{B}_{Y\times\mathbf{R}}$ spanned by v. Notice that the derivative of b along $\mathscr{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ is zero. We have the identification

$$T_A\mathscr{B}_{Y\times\mathbf{R}} \cong T_A\mathscr{B}^{\mathrm{bal}}_{Y\times\mathbf{R}} \times T_A N.$$

Since $B_{Y\times\mathbf{R}} \cong \mathscr{B}^{\mathrm{bal}}_{Y\times\mathbf{R}} \times \mathbf{R}$ and $D_t b(A) = \pm\mathrm{Id}$ in the time direction, we may identify $T_A N \cong T_0\mathbf{R} \cong (T_A\mathscr{B}_{Y\times\mathbf{R}})_t$, the last being the tangent space to $\mathscr{B}_{Y\times\mathbf{R}}$ at A in the time direction.

For $A \in \mathscr{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$, the cohomology H_A^1 is a 1-dimensional space. We claim that it contains $\{A(t+s)\colon s \in \mathbf{R}\}$. We have

$$H_A^1 = \{A(t) + sa(t)\colon s \in \mathbf{R},\ d_A^{*\delta} a = 0,\ d_A^+ a = 0\}.$$

Define $f(s, u) = A(t) + sa(t) - A(t-u)$. Then $f(0, 0) = 0$, and $\partial f(0, 0)/\partial u = A'(t) \neq 0$, since A is not a constant connection. Hence the implicit function theorem gives a local coordinate $u = u(s)$ in a neighborhood of $(0, 0)$ such that $f(s, u(s)) = 0$, i.e., $A(t) + sa(t) = A(t - u(s))$ in time-translation form. Let S be the subset of $\mathbf{R}$ defined by

$$S = \{s \in \mathbf{R}\colon \text{ there exists } u(s) \text{ such that } f(s, u(s)) = 0\}.$$

Then S is nonempty (since it contains 0), open (by the implicit function theorem) and closed (since $f(s, u(s))$ is continuous in s). Therefore $S = \mathbf{R}$, and $H_A^1 = \{A(t+s)\colon s \in \mathbf{R}\}$. Hence H_A^1 intersects $T_A\mathscr{B}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ transversely at the point $\{[A]\}$. The Kuranishi technique then implies that locally, solutions of the anti-self-duality equation live in a 1-dimensional moduli space parametrized by H_A^1, i.e., by time-translation.

3.2. Smallest eigenvalue on $Y \times \mathbf{R}$.

(i) Some analytical facts. Let d_A denote the covariant derivative corresponding to the connection A, and $d_A^{*\delta} = e_\delta^{-1} d_A^* e_\delta$ be the adjoint of d_A with respect to the $L^2_{0,\delta}$-norm. Floer has proved the following in [8].

Proposition 3.2.1 (Floer). (i) *For positive δ, $\mathscr{G}_\delta$ is a Banach Lie group with Lie algebra (which can be identified with) $L^p_{2,\delta}(\Omega^0_{\mathrm{ad}}(Y \times \mathbf{R}))$.*

(ii) *The quotient space* $\mathscr{B}_\delta(a, b) = \mathscr{A}_\delta^*(a, b)/\mathscr{G}_\delta$ *is a smooth Banach manifold with tangent spaces*

$$T_{[A]}\mathscr{B}_\delta(a, b) = \{\alpha \in L^p_{1,\delta}(\Omega^1_{\mathrm{ad}}(Y \times \mathbf{R})) | d_A^{*\delta}\alpha = 0\}.$$

(iii) *The 2-form* F_A^- *representing the anti-self-dual part of the curvature of* A *is smooth and* $\mathscr{G}_\delta$*-equivariant.*

(iv) *If* $\delta > 0$ *is smaller than the smallest nonzero absolute value of an eigenvalue of* D_a *or* D_b, *then for any anti-self-dual connection* $A \in \mathscr{B}_\delta(a, b)$ *the anti-self-duality operator*

$$D_A^\delta = d_A^{*\delta} \oplus d_A^+ : L^p_{1,\delta}\Omega^1_{\mathrm{ad}}(Y \times \mathbf{R}) \to L^p_{0,\delta}(\Omega^0_{\mathrm{ad}} \oplus \Omega^2_{\mathrm{ad},+})(Y \times \mathbf{R})$$

is Fredholm. Furthermore, $D_A^\delta = \partial/\partial t + D_{a_t}^\delta$, *where*

$$D_{a_t}^\delta = \begin{pmatrix} *d_a & -d_a \\ -d_a^* & \delta \end{pmatrix}.$$

$D_{a_t}^\delta$ *is self-adjoint on* $\Omega^1_{\mathrm{ad}}(Y) \otimes \Omega^0_{\mathrm{ad}}(Y)$, *and* $*$ *is the Hodge operator on the 3-manifold* Y. *If* a *and* b *are irreducible nondegenerate flat connections, then one can take* $\delta = 0$.

(v) *Let* $\mathscr{M}$ *be the moduli space of all equivalence classes of nonflat anti-self-dual connections* A *on* $Y \times \mathbf{R}$ *with finite action* $\|\partial A/\partial t\|_2^2$. *Then there is a first category set of metrics on* Y *such that the anti-self-duality operator* D_A^δ *is surjective for all* $A \in \mathscr{M} \cap \mathscr{B}_\delta$.

The following definitions are combined from [3], [4], [8] and [16].

Definition 3.2.2. An *ideal anti-self-dual connection* (*trajectory*) over $Y \times \mathbf{R}$, of Chern number k, is a pair

$$(A; (x_1, \cdots, x_l)) \in \mathscr{M}^{k-l}_{Y\times\mathbf{R}}(a, b) \times S^l(Y \times \mathbf{R}),$$

where A is a point in $\mathscr{M}^{k-l}_{Y\times\mathbf{R}}(a, b) \cap \mathscr{B}_\delta$, and $(x_1, \cdots, x_l)$ is an unordered l-tuple of points of $Y \times \mathbf{R}$.

Let $\{A_n\}$, $n \in \mathbf{N}$, be a sequence of connections of charge k on the SU(2) bundle P over $Y \times \mathbf{R}$. We say that the gauge equivalence classes $\{A_n\}$ *converge weakly* to a limiting ideal anti-self-dual connection $(A; (x_1, \cdots, x_l))$ if the following hold:

(i) The action densities converge as measures, i.e., for any continuous function on $Y \times \mathbf{R}$,

$$\int_{Y\times\mathbf{R}} f|F(A_n)|^2\, d\mu \to \int_{Y\times\mathbf{R}} f|F(A)|^2\, d\mu + 8\pi^2 \sum_{i=1}^{l} f(x_i).$$

(ii) There are bundle maps

$$\rho_n : P|_{Y\times\mathbf{R}\setminus\{x_1,\cdots,x_l\}} \to P|_{Y\times\mathbf{R}\setminus\{x_1,\cdots,x_l\}}$$

such that $\rho_n^*(A_n)$ converges to A in C^∞ on compact subsets of the punctured manifold.

Definition 3.2.3. Let a and b be flat SU(2) connections over Y. A *chain of connections* $(B_1, \cdots, B_n)$ from a to b is a finite set of connections over $Y\times\mathbf{R}$ which converge to flat connections c_{i-1}, c_i as $t\to\mp\infty$ such that $a=c_0$, $c_n=b$, and B_i connects c_{i-1}, c_i for $0\le i\le n$.

We say that the sequence $\{A_\alpha\}\in\mathscr{M}^k_{Y\times\mathbf{R}}(a, b)$ is (*weakly*) *convergent* to the chain of connections $(B_1, \cdots, B_n)$ if there is a sequence of n-tuples of real numbers $\{t_{\alpha,1}\le\cdots\le t_{\alpha,n}\}_\alpha$, such that $t_{\alpha,i}-t_{\alpha,i-1}\to\infty$ as $\alpha\to\infty$, and if, for each i, the translates $t^*_{\alpha,i}A_\alpha = A_\alpha(\circ - t_{\alpha,i})$ converge weakly to B_i.

We need to combine the notion of a chain connection with that of an ideal connection.

Definition 3.2.4. An *ideal chain connection* joining flat connections a and b over Y is a set $(A_j; x_{j1}, \cdots, x_{jl_j})_{1\le j\le J}$ where $(A_j)_{1\le j\le J}$ is a chain connection and for each j, $(A_j; x_{j1}, \cdots, x_{jl_j})$ is an ideal connection.

In this setup, a version of the Uhlenbeck compactness theorem holds. We state it in a form proved by Floer in [8] (see also [3]).

Theorem 3.2.5 (Uhlenbeck compactness on $Y\times\mathbf{R}$). *Let* $A_\alpha\in\mathscr{M}^k_{Y\times\mathbf{R}}\cap\mathscr{B}_\delta(a_\alpha, b_\alpha)$ *be a sequence of anti-self-dual connections with uniformly bounded action. Then there exists a subsequence converging to an ideal chain connection* $(A_j; x_{j1}, \cdots, x_{jl_j})_{1\le j\le J}$. *Moreover, one has*

$$\sum_{j=1}^{J}(k_j+l_j)=k, \qquad c_2(A_j)=k_j \ (\textit{not necessarily an integer}).$$

(ii) Smallest eigenvalue estimates.

(a) We will next prove the existence of a uniform bound for the eigenvalues of $\Delta_A^{2,+}$ for all A lying in 1-dimensional components of the moduli space.

Theorem 3.2.6. *Suppose* $\dim\mathscr{M}_{Y\times\mathbf{R}}=1$. *Then for each* $p\ge 2$ *there exists a positive constant* C_p *such that for all* $A\in\mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$, *and* $u\in L^p_{0,\delta}(\Omega^2_{\mathrm{ad},+}(Y\times\mathbf{R}))$ *we have*

$$C_p\cdot\int_{Y\times\mathbf{R}} e^{p\delta\cdot|t|}|u|^p \le \int_{Y\times\mathbf{R}} e^{p\delta\cdot|t|}|(d_A^+)^{*\delta}u|^p. \tag{10}$$

Proof. Proposition 3.2.1(v) implies that $(D_A^\delta)^*$ has trivial kernel. From the ellipticity of the anti-self-duality operator we have

$$C_{A,p}\|v \oplus u\|_{L^p_{1,\delta}} \leq \|(D_A^\delta)^*(v \oplus u)\|_{L^p_{0,\delta}}$$

for $v \oplus u \in (\Omega^0_{\mathrm{ad}} \oplus \Omega^2_{\mathrm{ad}+})(Y \times \mathbf{R})$. Thus, for any such $A \in \mathscr{M}^k_{Y\times\mathbf{R}} \cap \mathscr{B}_\delta$, by taking $p \geq 2$, $v = 0$, there is a positive real number $C_p(A)$ such that (10) holds for all $u \in \Omega^2_{\mathrm{ad},+}(Y \times \mathbf{R})$. Since the constant $C_p(A)$ is continuous in A, the theorem follows by Lemma 3.1.5.

Remark. The above estimate (10) also holds when A is the trivial connection. On $Y \times \mathbf{R}$, where Y is a fixed homology 3-sphere (with a fixed Riemannian metric), the standard Laplacian on the self-dual 2-forms has a strictly positive first eigenvalue by the Hodge Theorem, and the spectrum of such an operator consists of all the values which are greater than or equal to the first eigenvalue (see [9]). We can use this fact to get the bounded right inverse for d^+, and therefore we can glue a 1-dimensional trajectory on one side to a trivial connection on the other side.

(b) **The flattening construction.** We first describe a special gauge suited to our constructions. Fix $A \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$, and choose a trivialization of the fiber at a base pint $y \in Y$. Parallel transport first along the $\mathbf{R}$-direction, and then outward in normal coordinates in Y at each fixed time slice. This defines a gauge for $A \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ which we call the *cylindrical gauge.* In this gauge $A_t = 0$ on $\{y\} \times \mathbf{R}$, and $A_r = 0$ where r is the radius on Y centered at y.

Lemma 3.2.7. *In the cylindrical gauge in $B^3_y(\varepsilon)\times\mathbf{R}$, we have $|A(x,t)| \leq r\|F_A\|_\infty$.*

Proof. Let (x_1, x_2, x_3, t) be coordinates in $B^3_y(\varepsilon \times \mathbf{R}$. For $1 \leq i \leq 3$, we have $|A_i(x,t)| \leq (r/2)\max_{|(x,t)|<r}|F(x,t)|$ (c.f. [15]). Since $A_r = \sum_{k=1}^3 x_k A_k = 0$, we get $\sum_{k=1}^3 x_k(\partial A_k/\partial t) = 0$; thus

$$\sum_{k=1}^{3} x_k F_{kt} = r\frac{\partial}{\partial r}A_t - \sum_{k=1}^{3} x_k \frac{\partial A_k}{\partial t} = r\frac{\partial}{\partial r}A_t.$$

Also $\int_0^r \partial A_t/\partial r = A_t(x,t) - A_t(y,t) = A_t(x,t)$. Hence

$$|A_t(x,t)| \leq \left|\int_0^r \sum_{k=1}^{3} \frac{x_k}{r} F_{kt}\right| \leq r \max_{|(x,t)|<r} |F(x,t)|. \quad \text{q.e.d.}$$

We next need to describe how to flatten a connection $A \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ along $B_y(r_0) \times \mathbf{R}$. Let $\chi = \chi(r_0, \varepsilon)$ be a smooth cutoff function satisfying

$$\chi \equiv 0 \quad \text{on } B_y(r_0), \qquad \chi \equiv 1 \quad \text{on } Y\backslash B_y(r_0+\varepsilon), \qquad \text{and} \qquad |d\chi| \leq C_0/\varepsilon$$

for some constant C_0.

Definition 3.2.8. For $A \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$, define $\tilde{A} \in \mathscr{B}_{Y\times\mathbf{R}}$ to be the connection on E which is equal to A outside $B_y(r_0+\varepsilon)$ and $\tilde{A} = \chi \cdot A$ on $B_y(r_0+\varepsilon)$ as the connection matrix in the local trivialization of E given by the cylindrical gauge.

Lemma 3.2.9. *There exist ε_0 and C (independent of A) such that for $0<\varepsilon<\varepsilon_0$, any $A \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ with $\dim \mathscr{M}_{Y\times\mathbf{R}} \le 1$, and any $p, q \ge 2$,*

$$\|\tilde{A} - A\|_{L^q_{0,\delta}(Y\times\mathbf{R})} \le C\varepsilon^{(3+q)/q}, \qquad \|F^+_{\tilde{A}}\|_{L^p_{0,\delta}(Y\times\mathbf{R})} \le C\varepsilon^{3/p}.$$

Proof. Take $\chi = \chi(\varepsilon, \varepsilon)$ and $\tilde{A} = \chi \cdot A$. Then we have $F^+_{\tilde{A}} = (d\chi \wedge A)_+ + (\chi^2 - \chi)(A \wedge A)_+$ since A is anti-self-dual. $\tilde{A}$ has support on $B^3(2\varepsilon) \times \mathbf{R}$, and by Lemma 3.2.7 and Proposition 3.1.1, we have the pointwise bound

$$|F^+_{\tilde{A}}| \le C_0\varepsilon^{-1}|A| + |A|^2 \le C_0\varepsilon^{-1}2\varepsilon|F_A| + 4\varepsilon^2|F_A|^2 \le C_0'|F_A| \le C_0'Ce^{-\gamma|t|},$$

where $C_0'C$ is independent of A by Lemma 3.1.5. Hence

$$\|F^+_{\tilde{A}}\|_{L^p_{0,\delta}(Y\times\mathbf{R})} = \left(\int_{B^3(2\varepsilon)\times\mathbf{R}} |e^{\delta|t|}F^+_{\tilde{A}}|^p\right)^{1/p} \le C_2(\delta)\varepsilon^{3/p}.$$

The bound on $\tilde{A} - A$ is similar, namely,

$$|\tilde{A} - A| \le |A| \le 2\varepsilon|F_A| \le C\varepsilon e^{-\gamma|t|}.$$

Thus the result follows.

(c) **A neighborhood of $\mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$.** Assume throughout this subsection that the dimension of a moduli space satisfies $\dim \mathscr{M}_{Y\times\mathbf{R}} \le 1$. Fix $p, q > 2$. We are going to show that the uniform lower eigenvalue estimate also holds for nearby anti-self-dual connections.

Definition 3.2.10. Set

$$U_{\delta_1} = \{B \in \mathscr{B}_{Y\times\mathbf{R}} \mid \text{there exists a } A \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}} \text{ such that } \|A - B\|_{L^q_{0,\delta}} < \delta_1,\ \|F^+_B\|_{L^p_{0,\delta}} < \delta_1\}.$$

Note that Lemma 3.2.9 implies that if $A \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$, then for sufficiently small ε the flattened connection $\tilde{A}$ lies in U_{δ_1}.

Lemma 3.2.11. *There exists δ_0 such that for $0 < \delta_1 < \delta_0$ there is a constant C_5 independent of δ_1 such that*

$$\|u\|_{L^p_{1,\delta}(Y\times\mathbf{R})} \le C_5\|(d^+_B)^{*\delta}u\|_{L^p_{0,\delta}(Y\times\mathbf{R})} \quad \textit{for all } B \in U_{\delta_1}.$$

Proof. We have

$$\|(d_B^+)^{*\delta}u\|_{L^p_{0,\delta}Y\times\mathbf{R})} \geq \|(d_A^+)^{*\delta}u\|_{L^p_{0,\delta}(Y\times\mathbf{R})} - \|(A-B)*_\delta u\|_{L^p_{0,\delta}(Y\times\mathbf{R})},$$

where A is an element in $\mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}$ which is δ_1-close to B. Then

$$\|(A-B)*_\delta u\|_{L^p_{0,\delta}(Y\times\mathbf{R})} \leq \|A-B\|_{L^q_{0,\delta/2}}\|u\|_{L^4_{0,\delta/2}} \leq C\delta_1\|u\|_{L^p_{1,\delta}}$$

by Hölder's inequality and the weighted Sobolev embedding theorem [11] and [12].

Since A is anti-self-dual, the Weitzenböck formula gives $d_A^+(d_A^+)^{*\delta} = \nabla_A^{*\delta}\nabla_A + R$, which implies

$$\begin{aligned}\|u\|_{L^p_{1,\delta}} &\leq C_4\|u\|_{L^p_{1,\delta}(A)} \leq C_4C(p)\|(d_A^+)^{*\delta}u\|_{L^p_{0,\delta}} + C\|u\|_{L^p_{0,\delta}} \\ &\leq \widetilde{C}\|(d_A^+)^{*\delta}u\|_{L^p_{0,\delta}}.\end{aligned} \tag{11}$$

The first inequality follows from comparing $L^p_{1,\delta}$-norm and $L^p_{1,\delta}(A)$-norm, and the last from Theorem 3.2.6. Choosing δ_0 such that $C\widetilde{C}\delta_0 < \frac{1}{2}$, we have

$$\|(d_B^+)^{*\delta}u\|_{L^p_{0,\delta}} \geq \tfrac{1}{2}\|(d_A^+)^{*\delta}u\|_{L^p_{0,\delta}}. \tag{12}$$

Thus by (11) and (12), we obtain

$$\|u\|_{L^p_{1,\delta}} \leq \widetilde{C}\|(d_A^+)^{*\delta}u\|_{L^p_{0,\delta}} \leq 2\widetilde{C}\|(d_B^+)^{*\delta}u\|_{L^p_{0,\delta}}. \quad \text{q.e.d.}$$

By Lemma 3.2.11 and the weighted Sobolev embedding theorem [11], we have

$$L^p_{1,\delta} \hookrightarrow L^q_{0,\delta}, \quad \text{for } 1/4 + 1/q \geq 1/p,$$

and the bounded right inverse operator Q_B $(= (d_B^+)^{*\delta}(d_B^+(d_B^+)^{*\delta})^{-1})$ satisfies

$$\|Q_Bu\|_{L^q_{0,\delta}} \leq C\|Q_Bu\|_{L^p_{1,\delta}} \leq C\|u\|_{L^p_{0,\delta}} \quad \text{for all } B \in U_{\delta_1}.$$

Remark. If $\dim\mathscr{M}_{Y\times\mathbf{R}}(b_0, b_2) = 2$ and $\mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_0, b_2)$ is compact, then Theorem 3.2.6 and Lemma 3.2.11 are also true.

If $\mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_0, b_2)$ is not compact with boundary $\mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_0, b_1) \times \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_1, b_2)$, then for compact $K \subset \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_0, b_1) \times \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_1, b_2)$ and ρ_K large enough we get

$$A_0(\rho_K) = \chi(\rho_K)A_0, \qquad A_1(\rho_K) = \chi(-\rho_K)A_1$$

with $A_i(\rho_K) \in U_{\delta_1}(A_i)$, where $A_0 \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_0, b_1)$, $A_1 \in \mathscr{M}^{\mathrm{bal}}_{Y\times\mathbf{R}}(b_1, b_2)$, and χ is a smooth cutoff function on $Y\times\mathbf{R}$ satisfying $\chi(y, t) = 1$ and

0 for $t \leq \rho_K$ and $\geq \rho_K + 1$ respectively. Let $A_\chi = A_0(\rho_K)\hat{\#}A_1(\rho_K)$ on $Y \times \mathbf{R}$. Then Floer's Proposition 2d and inequality (2d.2) in [8] yield that $A_\chi \in U_{\delta_1}$ for very large ρ_K. Hence for $A_0\hat{\#}A_1$ we obtain the bounded right inverse by Lemma 3.2.11.

If $\dim \mathscr{M}_{Y\times\mathbf{R}}(b_0, b_4) \leq 4$, we get the bounded right inverse by using Proposition 1c.1 in [8] and the above argument. The dimension restriction comes from the trivial connection θ. If $\dim \mathscr{M}_{Y\times\mathbf{R}}(b_0, b_7) \leq 7$, then the moduli space does not split through θ, and Lemma 3.2.11 is also true for this case.

(d) **Changing metrics.** We want to show that there is also a bounded right inverse for flattened connections with metric C^0 close to the original metric. Pick a point $y_0 \in Y_0$. For simplicity we assume that the metric on Y_0 is flat in the 3-ball $B_3(r_0+\varepsilon)$ centered at y_0 with radius $r_0+\varepsilon$. For $r_1 < r_0$, let $N_{\varepsilon', r_0, r_1}(g_0)$ be the set of Riemannian metrics g on $Y_0 \backslash B_3(r_1)$ which satisfy

(i) $g = g_0$ on $Y_0 \backslash B_3(r_0)$;
(ii) $\|g - g_0\|_{C^0} < \varepsilon'$ on $B_3(r_0)\backslash B_3(r_1)$.

The annulus $B_3(r_0)\backslash B_3(r_1)$ will be used as the glueing region in forming connected sums.

(1) Let π_+^g be the projection onto self-dual 2-forms with respect to the metric g. Note that π_+^g is a continuous map with respect to the metrics, i.e., $\|\pi_+^g - \pi_+^{g_0}\| \leq C\|g - g_0\|_{C^0}$.

(2) For the metric g_0 on Y_0, there is a right inverse Q_0 for the operator $d_{\tilde{A}_0}^{+g_0}$. Let $S = d_{\tilde{A}_0}^{-g_0} Q_0$. Then

$$d_{\tilde{A}_0}^{+g_0} Q_0 u_0 = u_0, \quad \text{and} \quad \|Su_0\|_{L^p_{0,\delta}(g_0)} \leq C_p \|u_0\|_{L^p_{0,\delta}(g_0)},$$

where $\|\cdot\|_{L^p_{0,\delta}(g_0)}$ indicates the Sobolev space with metric g_0 for forms with support in $(Y_0 \backslash B_3(r_1)) \times \mathbf{R}$.

(3) For $g \in N_{\varepsilon', r_0, r_1}(g_0)$, the $L^p_{0,\delta}$-norms are equivalent, i.e.,

$$C_{\varepsilon'}^{-1} \|u_0\|_{L^p_{0,\delta}(g_0)} \leq \|u_0\|_{L^p_{0,\delta}(g)} \leq C_{\varepsilon'} \|u_0\|_{L^p_{0,\delta}(g_0)},$$

where $C_{\varepsilon'} \to 1$ as $\varepsilon' \to 0$.

Lemma 3.2.12. *For self-dual 2-forms u_0 with support in the $(Y_0\backslash B_3(r_1)) \times \mathbf{R}$ and $g \in N_{\varepsilon', r_0, r_1}(g_0)$ with sufficiently small ε', $d_{\tilde{A}_0}^{+g}$ has a right inverse Q_0^g with*

$$\|Q_0^g u\|_{L^p_{1,\delta}(g)} \leq C\|u\|_{L^p_{0,\delta}(g)},$$

and also

$$\|Q_0^g u\|_{L^q_{0,\delta}(g)} \le C\|u\|_{L^p_{0,\delta}(g)} \quad \text{for } 1/4 + 1/q \ge 1/p.$$

Proof. We will construct the right inverse by arranging that $d^{+_g}_{\tilde{A}_0} Q_0 - \mathrm{Id}$ is a contraction mapping on $L^p_{0,\delta}(g)(\Omega^2_+(Y_0 \times \mathbf{R}))$. We have

$$d^{+_g}_{\tilde{A}_0} Q_0 u_0 = d^{+_{g_0}}_{\tilde{A}_0} Q_0 u_0 + (d^{+_g}_{\tilde{A}_0} - d^{+_{g_0}}_{\tilde{A}_0}) Q_0 u_0,$$

and from (2) it follows that $(d^{+_g}_{\tilde{A}_0} Q_0 - \mathrm{Id})u_0 = (d^{+_g}_{\tilde{A}_0} - d^{+_{g_0}}_{\tilde{A}_0})Q_0 u_0$. By the definition of g and the flattening construction for A_0 with $\chi|_{[0,r_0]} \equiv 0$, one has

$$d^{+_g}_{\tilde{A}_0} - d^{+_{g_0}}_{\tilde{A}_0} = d^{+_g} - d^{+_{g_0}} = (\pi^g_+ - \pi^{g_0}_+)(d^{+_{g_0}} + d^{-_{g_0}}).$$

Using (1), (2), and (3) we obtain

$$\|(d^{+_g}_{\tilde{A}_0} Q_0 - \mathrm{Id})u_0\|_{L^p_{0,\delta}(g)} \le C C^2_{\varepsilon'} \varepsilon' (1 + C_p) \|u_0\|_{L^p_{0,\delta}(g)}.$$

For ε' so small that $CC^2_{\varepsilon'}\varepsilon'(1+C_p) < \frac{1}{2}$, the operator $d^{+_g}_{\tilde{A}_0} Q_0$ is invertible, and the right inverse for $d^{+_g}_{\tilde{A}_0}$ is $Q^g_0 = Q_0(d^{+_g}_{\tilde{A}_0} Q_0)^{-1}$. q.e.d.

For $B \in U_{\delta_1}$ and $g \in N_{\varepsilon', r_0, r_1}(g_0)$, we also get a bounded right inverse for the operator $d^{+_g}_B$ by combining the proofs of Lemmas 3.2.11 and 3.2.12.

3.3. Structure of the trajectory flow on the connected sum.

(i) Forming the connected sum.

(a) Let Y_i be an oriented homology 3-sphere with Riemannian metrics g_i, $i = 0, 1$. Choose basepoints $y_i \in Y_i$ and suppose for simplicity that the metrics g_i on Y_i are flat in neighborhoods of the y_i. Using these flat metrics we identify neighborhoods of the points y_i in Y_i with neighborhoods of zero in the tangent spaces $T_{y_i} Y_i$. Precisely, for any real numbers $\varepsilon, T > 0$, we set $N_{y_i}(\varepsilon, T) = \{(r, \theta): T^{-1}\varepsilon \le r \le T\varepsilon\} \subset T_{y_i} Y_i \backslash \{0\}$, where ε eventually will be made small and T (> 1) is another parameter (to be fixed later in the proof) with $T\varepsilon$ less than half the radius of injectivity of y_i. Then define $f_{\varepsilon,T}: N_{y_0}(\varepsilon, T) \to N_{y_1}(\varepsilon, T)$ by $f_{\varepsilon,T}(r, \theta) = (-r + T\varepsilon + T^{-1}\varepsilon, \theta)$. Let $U_i \subset Y_i$ be the annulus centered at y_i with inner radius $r_1 = T^{-1}\varepsilon$ and outer radius $r_0 = T\varepsilon$. The "linear inversion" map $f_{\varepsilon,T}$ taking the inner radius of U_0 to the outer radius of U_1 induces an orientation-reversing diffeomorphism from U_0 to U_1. Let

$Y_i' \subset Y_i$ be the open set obtained by removing the $T^{-1}\varepsilon$ ball about y_i. Then, in the usual sense, we define the connected sum $Y = Y(\varepsilon, T)$ to be

$$Y = Y_0 \# Y_1 = Y_0' \cup_{f_{\varepsilon, T}} Y_1',$$

where the annuli U_i are identified by $f_{\varepsilon, T}$.

(b) Let (Y_i, g_i), $i = 0, 1$, be oriented Riemannian 3-manifolds as in (a). To construct a Riemannian metric on $Y_0 \# Y_1$, we fix a cutoff function, $\phi \in C^\infty([0, +\infty))$, which satisfies

$$\phi|_{[0, T^{-1}\varepsilon]} \equiv 0, \qquad \phi\left(\frac{T + T^{-1}}{2}\varepsilon\right) = \frac{1}{2}, \qquad \text{and} \quad \phi|_{[T\varepsilon, +\infty)} \equiv 1.$$

Definition 3.3.1. The *Riemannian metric* g on the connected sum $Y_0 \# Y_1$ is defined as follows: On $Y_i \backslash B_{y_i}(T\varepsilon)$, set $g = g_i$ for $i = 0, 1$. On the overlap annulus $N_{y_0}(\varepsilon, T) \cong N_{y_1}(\varepsilon, T)$, $g = \phi g_0 + (1 - \phi) f_{\varepsilon, T}^* g_1 = \phi g_0 + f_{\varepsilon, T}^*(\phi g_1)$, because of the linearity of $f_{\varepsilon, T}$.

Lemma 3.3.2. *Let ε' be the constant of Lemma* 3.2.12. *There exists $T_0 > 1$ such that for all $1 < T \leq T_0$, $\exists \varepsilon$ with $T\varepsilon < \frac{1}{2}\{$injectivity radius$\}$, and we have $N_{\varepsilon', T\varepsilon, T^{-1}\varepsilon}(g_i) \neq \varnothing$. Furthermore, $N_{\varepsilon', T\varepsilon, T^{-1}\varepsilon}(g_0) \cap N_{\varepsilon', T\varepsilon, T^{-1}\varepsilon}(g_1) \neq \varnothing$.*

Proof. We just use the metric from Definition 3.3.1 and calculate the C^0 norm of $g - g_0$ on the annular region:

$$g_{rr} = (g_0)_{rr}, \quad \text{and} \quad g_{\theta\theta} = \left\{\phi + (1 - \phi)\left(-1 + \frac{(T + T^{-1})\varepsilon}{r}\right)^2\right\}(g_0)_{\theta\theta}.$$

Then $\forall T \leq T_0$, we have

$$\|g - g_0\|_{C^0} \leq \max\{T^4 - 1, |T^{-4} - 1|\}.$$

By choosing T_0 close to 1 enough to make $\|g - g_0\|_{C^0} \leq \varepsilon'$, we thus prove the lemma.

Remark. Lemma 3.3.2 tells us that we may glue the two manifolds by an orientation-reversing isometry on the tiny overlap region. Therefore for forms u supported on Y_i', we have

$$\tfrac{1}{2}\|u\|_{L^p_{0,\delta}(g_i)(Y_i')} \leq \|u\|_{L^p_{0,\delta}(g)(Y_0 \# Y_1)} \leq 2\|u\|_{L^p_{0,\delta}(g_i)(Y_i')}.$$

(c) We next use the SU(2)-bundles P_i over Y_i to define a bundle P over Y. Using the projection map $\pi_1 \colon Y_i \times \mathbf{R} \to Y_i$, we pull back the bundles P_i to get bundles $\pi_1^*(P_i)$ over $Y_i \times \mathbf{R}$. Let A_0 be a flat connection on $Y_0 \times \mathbf{R}$, constant in the sense that $A_0(t) = \alpha \in \mathscr{R}(Y_0)$ for all $t \in \mathbf{R}$, and let A_1 be an anti-self-dual trajectory from β to γ (i.e., an anti-self-dual connection lying in a 1-dimensional moduli space) on $Y_1 \times \mathbf{R}$. Set $\tilde{A}_i = \chi A_i$, $i = 0, 1$ by using the flattening procedure on each side as in §3.2(ii)(b) with $\chi = \chi(T\varepsilon, \varepsilon)$.

Choose an SU(2)-isomorphism of the fibers $\rho: (P_0)_{y_0} \to (P_1)_{y_1}$. Using the flat structures $\tilde{A}_i$, both of which are flat on the overlap, we can spread out this isomorphism by parallel transport to give a bundle isomorphism g_ρ between the P_i over the identified part (an annulus or conformally spherical tube) covering $f_{\varepsilon, T}$. We call such a bundle isomorphism g_ρ a *glueing map.* Use this glueing map to construct a bundle $P_0 \cup_\rho P_1$ over $Y = Y_0 \#_{\varepsilon, T} Y_1$ and also the pullback bundle $\pi_1^*(P_0 \cup_\rho P_1) = E(\rho)$ over $Y \times \mathbf{R}$. The glueing map g_ρ is referred to the connections $\tilde{A}_i$, so we get an induced connection $A_\rho = A_0 \#_\rho A_1$ on $E(\rho)$. Thus $A_\rho = A_0 \#_\rho A_1$ is equal to $\tilde{A}_i$ on $(Y_i \backslash B_3(T^{-1}\varepsilon)) \times \mathbf{R}$. Note that A_ρ is trivial over the region identified by the glueing map.

The connections A_ρ, for different ρ, are not in general gauge equivalent, even though the bundles $E(\rho)$ are obviously isomorphic. Let Γ_{A_i} be the isotropy group of A_i over $Y_i \times \mathbf{R}$ and let $\Gamma = \Gamma_{A_0} \times \Gamma_{A_1}$. The equivalence classes of connections constructed in this way are in one-to-one correspondence with

$$\mathrm{Hom}_{\mathrm{SU}(2)}((P_0)_{y_0}, (P_1)_{y_1}) = \mathrm{SU}(2)/\Gamma,$$

the space of "glueing parameters". When the A_i are irreducible, $\Gamma = \{\pm 1\}$ so the space of glueing parameters is SO(3).

The following proposition can be found in the text of Donaldson and Kronheimer [3, p. 286].

Proposition 3.3.3. *The connections A_{ρ_1}, A_{ρ_2} are gauge equivalent if and only if the parameters ρ_1, ρ_2 are in the same orbit of the action of Γ on* SU(2).

The following proposition follows from Lemmas 3.3.2 and 3.2.12, by recalling the constants ε_0 of Lemma 3.2.9 and T_0 of Lemma 3.3.2.

Proposition 3.3.4. *For $0 < \varepsilon < \varepsilon_0$ and $1 < T < T_0$, there is a constant C independent of ε such that the operator $d_{A_\rho}^{+g}$ has a bounded right inverse G with*

$$\|Gu\|_{L^p_{1,\delta}(g)(Y_0 \#_{\varepsilon, T} Y_1)} \le C \|u\|_{L^p_{0,\delta}(g)(Y_0 \#_{\varepsilon, T} Y_1)}$$

and

$$\|Gu\|_{L^q_{0,\delta}(g)} \le C \|u\|_{L^p_{0,\delta}(g)}, \quad 1/4 + 1/q \ge 1/p.$$

Proof. The right inverse for $d_{A_i}^{+g}$ is Q_i^g. Then using the definition of A_ρ, we define $G'u = \chi_0 Q_0^g u_0 + \chi_1 Q_1^g u_1$, where $u_0 = (1-\eta_1)u$, $u_1 = \eta_1 u$, and η_1 is a smooth cutoff function on the annulus $U_0 \cap U_1$ which obeys $\eta_1|_{[0,T^{-1}\varepsilon]} = 0$ and $\eta_1|_{\{T\varepsilon \le r\}} = 1$, $\chi_0|_{[0,T\varepsilon]} = 1$, $\chi_0|_{\{r \ge 3T\varepsilon/2\}} = 0$ and $\chi_1|_{[0,T^{-1}\varepsilon/2]} = 0$, $\chi_1|_{\{r \ge T\varepsilon\}} = 1$. Since $\chi_i = 1$ on the support of η_i, G' is a right parametrix and $d_{A_\rho}^{+g} G' - \mathrm{Id}$ has a C^∞-kernel, it follows that $G = G'(d_{A_\rho}^{+g} G')^{-1}$ has the desired properties.

Remark. Proposition 3.3.4 also holds for $\dim \mathscr{M}_{Y_i \times \mathbf{R}} \le 4$ by the remark after Lemma 3.2.11. It is also true for $\dim \mathscr{M}_{Y_i \times \mathbf{R}} \le 7$ if $\mathscr{M}_{y_i \times \mathbf{R}}$ does not split through the trivial connection.

(ii) Gluing and splitting. Our goal is to deform the "almost anti-self-dual" connection A_ρ to a nearby anti-self-dual connection $A_\rho + a_\rho$. This entails solving the nonlinear anti-self-duality equation

$$F^+(A_\rho) + d_{A_\rho}^+ a + (a \wedge a)_+ = 0.$$

The upshot of Proposition 3.3.4 is that we are able to solve the linearized anti-self-duality equation $d_A^+ u = b$ over $Y = Y_0 \#_{\varepsilon, T} Y_1$, as long as A is irreducible (i.e., $H_A^0 = 0$) and regular (i.e., $H_A^2 = 0$), and futhermore there are estimates on the solution of the corresponding linearized equation which are independent of ε. We shall use the inverse function theorem to deform the almost anti-self-dual connection A_ρ.

Lemma 3.3.5 (cf. [8]). *Let $f\colon E \to F$ be a C^1 map between Banach spaces. Assume that in the first order Taylor expansion $f(\xi) = f(0) + Df(0)\xi + N(\xi)$, $Df(0)$ has a finite dimensional kernel and a right inverse G such that for $\xi, \zeta \in E$*

$$\|GN(\xi) - GN(\zeta)\|_E \le C(\|\xi\|_E + \|\zeta\|_E)\|\xi - \zeta\|_E$$

for some constant C. Let $\delta_1 = (8C)^{-1}$. If $\|Gf(0)\|_E \le \delta_1/3$, then there exists a C^1-function $\phi\colon K_{\delta_1} \to \operatorname{Im} G$ with $f(\xi + \phi(\xi)) = 0$ for all $\xi \in K_{\delta_1}$, and furthermore we have the estimate

$$\|\phi(\xi)\|_E \le \tfrac{4}{3}\|Gf(0)\|_E + \tfrac{1}{3}\|\xi\|_E,$$

where $K_{\delta_1} = \operatorname{Ker} Df(0) \cap \{\xi \in E\colon \|\xi\|_E < \delta_1\}$.

Applying Lemma 3.3.5 to $f(a) = F^+(A_\rho) + d_{A_\rho}^+ a + (a \wedge a)_+$ with $f(0) = F^+(A_\rho)$, $N(a) = (a \wedge a)_+$, $Df(0) = d_{A_\rho}^+$ (with the bounded right inverse G from the Proposition 3.3.4), $E = L_{1,\delta}^p \cap L_{0,\delta}^q(T_{A_\rho}\mathscr{B})$ and $F = L_{0,\delta}^p(\Omega_+^2(Y \times \mathbf{R}, \mathrm{ad}))$, we have the following theorem.

Theorem 3.3.6. *Let Y_i $(i = 0, 1)$ be homology 3-sphere and $A_i \in \mathscr{M}^{\text{bal}}_{Y_i \times \mathbf{R}}$. Assume $\dim \mathscr{M}_{Y_0 \times R} = 0$ and $\dim \mathscr{M}_{Y_1 \times \mathbf{R}} = 1$. Let ε_0 be the constant of Lemma 3.2.9. If $0 < \varepsilon < \varepsilon_0$ and $1 < T < T_0$ which is chosen from Lemma 3.3.2, then we can deform A_p to a smooth anti-self-dual connection over $(Y_0 \#_{\varepsilon, T} Y_1) \times \mathbf{R}$.*

Proof. Using Proposition 3.3.4 and Lemma 3.2.9, we obtain

$$\begin{aligned}\|GF^+(A_p)\|_{L^q_{0,\delta}} &\le C\|F^+(A_p)\|_{L^p_{0,\delta}} \\ &\le C(\|F^+(\tilde{A}_0)\|_{L^p_{0,\delta}} + \|F^+(\tilde{A}_1)\|_{L^p_{0,\delta}}) \le C_3 \varepsilon^{3/p}\end{aligned}$$

and $N(a) - N(b) = ((a-b) \wedge a)_+ (b \wedge (a-b))_+$. On the other hand, the weighted Hölder inequality and Lemma 7.2 in [12]

$$\|((a-b)\wedge a)_+\|_{L^p_{0,\delta}} \le \|a-b\|_{L^q_{0,\delta/2}} \|a\|_{L^4_{0,\delta/2}} \le C_\delta \|a-b\|_{L^q_{0,\delta}} \|a\|_{L^q_{0,\delta}},$$

where $C_\delta = c(\mathrm{Vol}(Y_0) + \mathrm{Vol}(Y_1) + 1)/\delta$. So

$$\|GN(a) - GN(b)\|_{L^q_{0,\delta}} \le CC_\delta \|a-b\|_{L^q_{0,\delta}} (\|a\|_{L^q_{0,\delta}} + \|b\|_{L^q_{0,\delta}}),$$

and by Lemma 3.3.5 with $\delta_1 = (8CC_\delta)^{-1}$, there exists $\phi\colon H^1_{A_p} \to \operatorname{Im} G$ with $f(\xi + \phi(\xi)) = 0$, where $\phi(A_p) = a_p$. Thus $A_p + a_p$ is an ASD connection over $(Y_0 \#_{\varepsilon, T} Y_1) \times \mathbf{R}$ with $\|a_p\|_{L^q_{0,\delta}}$ small, and is smooth by standard elliptic regularity.

Remarks. (i) The restriction on dimensions of moduli spaces comes form Lemma 3.2.9 and Proposition 3.3.4, in order to be able to get the bounded right inverse. Also from the proof above it is easy to see that we can glue two 1-dimensional anti-self-dual connections into a 2-dimensional anti-self-dual connection.

(ii) Using the remark after Theorem 3.2.6 and the construction in Proposition 3.3.4, we can also deform the $A_0 \# A_1$ into the anti-self-dual connection when one of A_i is trivial.

(iii) Combining the remarks after Proposition 3.3.4 and Theorem 3.3.6 we can deform the $A_0 \# A_1$ into anti-self-dual connection when $A_i \in \mathscr{M}^{\text{bal}}_{Y_i \times \mathbf{R}}$ with dimension ≤ 4. (For dimension ≤ 7 we need A_i which do not split through θ.)

To incorporate the glueing parameter SO(3), we apply the parametrized version of Lemma 3.3.5 which states that the solution depends smoothly on the parameters and is well-behaved under gauge transformations. That gives the description of a model for an open subset in the moduli space $\mathscr{M}^1_{Y_0 \# Y_1 \times \mathbf{R}}$.

Theorem 3.3.7. *Given a constant flat anti-self-dual connection A_0 and a 1-dimensional anti-self-dual connection A_1 with each D_{A_i} surjective in the weighted Sobolev space, and for small enough ε and all glueing parameters ρ, there is a smooth anti-self-dual connection $(A_0\#_\rho A_1)+a_\rho(t)$. If ρ_1, ρ_2 are in the same orbit under the Γ action on the space of glueing parameters* $SU(2)$, *the corresponding anti-self-dual connections are gauge equivalent.*

The restrictions on ε and T imposed on Theorem 3.3.6 mean that the "neck" region of the connected sum must be narrow with very small radius. Conversely, when our metric satisfies these conditions, we can characterize the anti-self-dual solutions found by our glueing construction. Define

$$Gl_\varepsilon : \mathscr{M}^{i_0}_{Y_0\times\mathbf{R}} \times SO(3) \times \mathscr{M}^{i_1}_{Y_1\times\mathbf{R}} \to \mathscr{B}_{(Y_0\#Y_1)\times\mathbf{R}}$$

by $Gl_\varepsilon(A_0, \rho, A_1) = A_0\#_\rho A_1$ as in §3.3(i)(c), where $i_0 \geq 0$, $i_1 \geq 0$, $i_0 + i_1 = 1$. Now for $\delta_2 > 0$ in the proof of Theorem 3.3.6, let $U_{\delta_2}(\varepsilon) \subset \mathscr{B}_{Y\times\mathbf{R}}$ be the open set

$$U_{\delta_2}(\varepsilon) = \{A| \inf_{B\in \operatorname{Im} Gl_\varepsilon} \|A-B\|_{L^q_{0,\delta}(g)((Y_0\#_{\varepsilon,T}Y_1)\times\mathbf{R})} < \delta_2,$$
$$\|F_A^+\|_{L^p_{0,\delta}(g)((Y_0\#_{\varepsilon,T}Y_1)\times\mathbf{R})} < \delta_2\}.$$

The solutions to the anti-self-duality equation obtained from Theorem 3.3.6 lie in $U_{\delta_2}(\varepsilon)$, and any element in U_{δ_2} can be deformed to a unique anti-self-dual connection by Lemma 3.3.5 (the uniqueness follows from the contraction mapping principle on $T\mathscr{B}^{\text{bal}}_{Y\times\mathbf{R}}$).

Theorem 3.3.8. *For ε, T as in Theorem* 3.3.6, *any point in $U_{\delta_2}(\varepsilon) \cap \mathscr{M}^1_{Y_0\#Y_1\times\mathbf{R}}(g_\varepsilon)$ can be represented by a connection A of the form $A_0\#_\rho A_1 + \phi(A_0\#_\rho A_1)$, where A_i is a* 0- *or* 1-*dimensional anti-self-dual connection on $\mathscr{M}_{Y_i\times\mathbf{R}}$, and ϕ is the C^1-diffeomorphism in the proof of Theorem* 3.3.6 *with $\|\phi(A_0\#_\rho A_1)\|_{L^q_{0,\delta}} < \delta_2$.*

Proof. Assume the contrary. Then there exists a sequence $\varepsilon_n \to 0$ with $\varepsilon_n < \varepsilon_0$, $\{[A_n]\} \in U^c_{\delta_2}(\varepsilon_n) \cap \mathscr{M}^1_{Y_0\#Y_1\times\mathbf{R}}(g_{\varepsilon_n})$ where $U^c_{\delta_2}(\varepsilon_n)$ is complement of U_{δ_2}, i.e., the A_n are not of this form.

By Uhlenbeck's compactness theorem applied to the balanced anti-self-dual connections, we have a subsequence converging to $A_0 \vee A_1$, where A_i is an anti-self-dual connection on $(Y_i\backslash\{y_i\}) \times \mathbf{R}$, since 1-dimensional moduli space is compact up to time-translation by Lemma 3.1.5. The connection A_i has a singularity along a line $\{y_i\}\times\mathbf{R}$. Since the singularity $\{y_i\} \times \mathbf{R}$ is codimension 3, it can be removed by Sibner's theorem [13]. Let the extended anti-self-dual connections still be denoted by A_i. By the flattening construction, for small enough ε_n we obtain

$$\tilde{A}_i(\varepsilon_n) = \chi(T\varepsilon_n, \varepsilon_n)A_i \quad \text{with } \|\tilde{A}_i(\varepsilon_n) - A_i\|_{L^q_{0,\delta}(g_n)(Y_i\times\mathbf{R})} < \delta_2/8.$$

Let $A_\rho(\varepsilon_n) = A_0\#_{\rho,\varepsilon_n}A_1$ as in §3.3(i)(c). Then

$$\begin{aligned}
&\|A_n - A_\rho(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_0\#_{\varepsilon_n,T}Y_1)\times\mathbf{R})} \\
&\le \|A_n - \tilde{A}_0(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_0\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})} \\
&\qquad + \|A_n - \tilde{A}_1(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_1\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})}, \\
&\le \sum_{i=0}^{1}\{\|A_n - A_i\|_{L^q_{0,\delta}(g_n)((Y_i\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})} \\
&\qquad + \|\tilde{A}_i(\varepsilon_n) - A_i\|_{L^q_{0,\delta}(g_n)((Y_i\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})}\}.
\end{aligned}$$

For n large enough we have $\|A_n - A_i\|_{L^q_{0,\delta}(g_n)((Y_i\backslash B_3(T^{-1}\varepsilon_n))\times\mathbf{R})} < \delta_2/8$ by convergence. Thus

$$\|A_n - A_\rho(\varepsilon_n)\|_{L^q_{0,\delta}(g_n)((Y_0\#_{\varepsilon_n,T}Y_1)\times\mathbf{R})} < \delta_2/2.$$

Since $A_n \in \mathcal{M}^1_{(Y_0\#_{\varepsilon_n,T}Y_1)\times\mathbf{R}}(g_n)$, of course $F^+(A_n) = 0$, so $A_n \in U_{\delta_2/2}(\varepsilon_n)$ which contradicts $A_n \in U^c_{\delta_2}(\varepsilon_n)$.

Thus for sufficiently small ε, a 1-dimensional moduli space can be represented by the one deformed from the glueing process.

Corollary 3.3.9. *Under the assumption of Theorem* 3.3.8, *there is a unique small solution to the anti-self-duality equation. So* $U_{\delta_2}(\varepsilon)\cap\mathcal{M}^1_{Y_0\#Y_1\times\mathbf{R}}$ *is equal to the image of the glueing map.*

This is the main analytic result of this paper. We summarize this subsection in the following theorem.

Theorem 3.3.10. *Suppose* A_i *is an anti-self-dual connection on* $Y_i\times\mathbf{R}$, *and consider the connected sums* $Z = (Y_0\#_{\varepsilon,T}Y_1)\times\mathbf{R}$ *for fixed* $0<\varepsilon<\varepsilon_0$, $1<T<T_0$. *Then for sufficiently small* ε *and each* $g\in SO(3)$ *one has the map*

$$\bigcup_{i+i'=1}\mathcal{M}^1_{Y_i\times\mathbf{R}}(\alpha_i, \beta_i)\times\mathcal{M}^0_{Y_{i'}\times\mathbf{R}}(\gamma_{i'}, \gamma_{i'}) \overset{\text{glueing}}{\longrightarrow} \mathcal{M}^1_Z(\alpha_i\#_g\gamma_{i'}, \beta_i\#_g\gamma_{i'}).$$

Conversely any ASD connections in $\mathcal{M}^1_Z(\alpha_i\#_g\gamma_{i'}, \beta_{i\#_g}\gamma_{i'})$ *can be obtained in this way.*

For $\dim\mathcal{M}_{Y_i\times\mathbf{R}}\le 4$ the glueing from $Y_i\times\mathbf{R}$ and the glueing from the connected sum give all the anti-self-dual connections on $(Y_0\#Y_1)\times\mathbf{R}$.

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